

Complex Dunkl operators ^{*}

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Abstract

The real theory of the Dunkl operators has been developed very extensively, while there still lacks the corresponding complex theory. In this paper we introduce the complex Dunkl operators for certain Coxeter groups. These complex Dunkl operators have the commutative property, which makes it possible to establish a corresponding complex analysis of Dunkl operators.

Keywords: Dunkl operators, complex Coxeter group

1 Introduction

In the real Euclidean spaces, Dunkl extended the classical real analysis by introducing the generalized differential operators related to Coxeter groups

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[3, 4]. The resulting operators are a family of commutative differential-difference operators \mathcal{T}_j , called Dunkl operators, which can be considered as perturbations of the usual partial derivatives by reflection parts. These operators step from the analysis of quantum many body system of Calogero-Moser-Sutherland type [2] in mathematical physics. They also have root in the theory of special functions of several variables, motivated from the theory of Riemann symmetric spaces, whose spherical functions can be written as multi-variable parametrized special functions. We refer to [1, 6, 8, 9, 10] for the theory of special functions, Fourier transforms, Segal-Bargmann transforms, and Clifford analysis in the Dunkl setting.

The purpose of the article is to establish the basis for the complex Dunkl analysis by introducing the complex Dunkl operators for certain Coxeter groups and establishing their important property of commutativity. We mention that Dunkl and Opdam [5] also introduced the Dunkl operators for complex reflection groups. But their operators are generalization of the real partial differentiate operators, other than the generalization of the complex operator $\frac{\partial}{\partial z}$.

2 Real Dunkl operators

In this section, we recall the definition of the real Dunkl operators.

In \mathbb{R}^N , we consider the standard inner product

$$\langle x, y \rangle = \sum_{j=1}^N x_j y_j$$

and the norm $\|x\| = \langle x, x \rangle^{1/2}$.

For a row vector $u \in \mathbb{R}^N \setminus \{0\}$, the reflection in the hyperplane $\langle u \rangle^\perp$ orthogonal to u is defined by

$$\sigma_u(x) = x\sigma_u := x - 2 \frac{\langle x, u \rangle}{\|u\|^2} u.$$

A root system is a finite set R of nonzero vectors in \mathbb{R}^N such that

$$\sigma_u(R) = R, \quad \text{and} \quad R \cap \mathbb{R}u = \{\pm u\}$$

for any $u \in R$.

A positive subsystem R_+ is any subset of R satisfying $R = R_+ \cup \{-R_+\}$. For example, for any $u_0 \in \mathbb{R}^N$ such that $\langle u, u_0 \rangle \neq 0$ for all $u \in R$, then $R_+ := \{u \in R : \langle u, u_0 \rangle > 0\}$ is a positive subsystem.

The real Coxeter group $W = W(R)$ (or real finite reflection group) generated by the root system $R \subset \mathbb{R}^N$ is the subgroup of orthogonal group $O(N)$ generated by $\{\sigma_u : u \in R\}$.

A multiplicity function on R is a complex-valued function

$$\begin{aligned} \kappa : R &\longrightarrow \mathbb{C} \\ v &\longmapsto \kappa_v \end{aligned}$$

which is invariant under the Coxeter group, i.e.,

$$\kappa_u = \kappa_{ug}, \quad \forall u \in R, \quad \forall g \in W.$$

The Dunkl operator \mathcal{T}_j , associated with the Coxeter group $W(R)$ and the multiplicity function κ , is the first order differential-difference operator:

$$\mathcal{T}_j p(x) = \frac{\partial p(x)}{\partial x_j} + \sum_{v \in R_+} \kappa_v \frac{p(x) - p(x\sigma_v)}{\langle x, v \rangle} v_j$$

for any polynomials p in \mathbb{R}^N .

The Dunkl operator \mathcal{T}_j is a homogeneous differential operator of degree -1 . By the W -invariance of the multiplicity function κ , we have

$$g^{-1} \circ \mathcal{T}_u \circ g = \mathcal{T}_{ug}, \quad \forall g \in W(R), \quad u \in \mathbb{R}^N,$$

where

$$\mathcal{T}_u = \sum_{j=1}^N u_j \mathcal{T}_j.$$

The remarkable property of the Dunkl operators is that the family $\{\mathcal{T}_u, u \in \mathbb{R}^N\}$ generates a commutative algebra of linear operators on the \mathbb{C} -algebra of polynomial functions on \mathbb{R}^N .

Example 2.1 *In the one-dimensional case $N = 1$, the root system R is of type A_1 (see [7]), the reflection group $W = \mathbb{Z}_2$, and the multiplicity function is given by a single parameter $\kappa \in \mathbb{C}$. The Dunkl operator is given by*

$$\mathcal{T}f(x) = f'(x) + \kappa \frac{f(x) - f(-x)}{x}.$$

3 Complex Dunkl operators

Assumption A: Under the standard embedding $\mathbb{R}^N \subset \mathbb{C}^N$, we assume that R is a root system in $\mathbb{C}^N \cong \mathbb{R}^{2N}$ satisfying

$$R \subset \mathbb{R}^N.$$

Let R be a root system in \mathbb{C}^N satisfying Assumption A. Then $R \subset \mathbb{R}^N$, so that we can define the positive subsystem $R_+ \subset \mathbb{R}^N \subset \mathbb{C}^N$.

For any operator $A \in O(N)$, we can regard A as a real orthogonal matrix under a fixed basis $\{e_\alpha\}$ of \mathbb{R}^N . Therefore, A is also a unitary matrix in \mathbb{C}^N . Under the same basis $\{e_\alpha\}$ of \mathbb{C}^N , the matrix A can thus be regarded as a unitary operator in \mathbb{C}^N . Therefore the matrix point of view results in a standard embedding

$$O(N) \hookrightarrow U(N),$$

where $U(N)$ is the unitary group.

In \mathbb{C}^N , we will use the same notation $\langle \cdot, \cdot \rangle$ for the extension of the Euclidean inner product, i.e.,

$$\langle z, w \rangle = \sum_{j=1}^N z_j \overline{w_j}, \quad \forall z, w \in \mathbb{C}^N,$$

and denote the norm $\|z\| = \langle z, z \rangle^{1/2}$. We also denote the bilinear extension of the Euclidean inner product by

$$(z, w) = \sum_{j=1}^N z_j w_j.$$

Let

$$v \in R \subset \mathbb{R}^N \hookrightarrow \mathbb{C}^N$$

and define the reflection $\sigma_v \in U(N)$ as matrix by

$$(\sigma_v)_{ij} = \delta_{ij} - 2 \frac{v_i v_j}{\|v\|^2},$$

so that

$$\sigma_v(z) = z \sigma_v := z - 2 \frac{\langle z, v \rangle}{\|v\|^2} v = z - 2 \frac{(z, v)}{\|v\|^2} v.$$

The complex Coxeter group $G = G(R)$ is the subgroup of $U(N)$ generated by $\{\sigma_v : v \in R\}$, namely,

$$G := \langle \sigma_v \in U(N) : v \in R \subset \mathbb{R}^N \hookrightarrow \mathbb{C}^N \rangle.$$

Consequently, for our choice of the root system, we have

$$G \subset O(N) \hookrightarrow U(N).$$

A multiplicity function κ is a G -invariant complex-valued function on R .

Let $R \subset \mathbb{R}^N$ be a root system and R_+ be a positive subsystem. Now we define the complex Dunkl operator as the first order differential-difference operators in coordinate form for $1 \leq j \leq N$ by

$$\mathcal{D}_j p(z) = \frac{\partial p(z)}{\partial z_j} + \sum_{v \in R_+} \kappa_v \frac{p(z) - p(z\sigma_v)}{\langle z, v \rangle} v_j$$

for any holomorphic polynomials p in \mathbb{C}^N .

To understand the second summand, notice that

$$\frac{dp}{dt} = \sum_{j=1}^N \frac{dp}{dz_j} \frac{dz_j}{dt} + \sum_{j=1}^N \frac{dp}{d\bar{z}_j} \frac{d\bar{z}_j}{dt} \quad \text{and} \quad \frac{dp}{d\bar{z}_j} = 0,$$

we have

$$\begin{aligned} \frac{p(z) - p(z\sigma_v)}{\langle z, v \rangle} &= -\frac{1}{\langle z, v \rangle} \int_0^1 \frac{dp}{dt} (t(z\sigma_v) + (1-t)z) dt \\ &= \int_0^1 \left\langle \nabla p(t(z\sigma_v) + (1-t)z), \frac{2v}{\|v\|^2} \right\rangle dt, \end{aligned}$$

where ∇ is the complex gradient

$$\nabla p(z) = \left(\frac{\partial P(z)}{\partial z_1}, \dots, \frac{\partial P(z)}{\partial z_N} \right).$$

For the coordinate free form, let $u \in \mathbb{C}^N$ and p be a polynomial in \mathbb{C}^N . We define the complex Dunkl operator by

$$\mathcal{D}_u p(z) = \sum_{j=1}^N \bar{u}_j \mathcal{D}_j p(z),$$

i.e.,

$$\mathcal{D}_u p(z) = \langle \nabla p(z), u \rangle + \sum_{v \in R_+} \kappa_v \frac{p(z) - p(z\sigma_v)}{\langle z, v \rangle} \langle v, u \rangle. \quad (3.1)$$

4 Some lemmas

For the proof of our main theorem, we need some lemmas.

The collection of all polynomials in z is denoted by $\mathbb{C}[z_1, \dots, z_N]$. We shall also use the abbreviation

$$\Pi^N := \mathbb{C}[z_1, \dots, z_N].$$

The right regular representation of $U(N)$ is the homomorphism

$$\begin{aligned} \mathcal{R} : U(N) &\longrightarrow \text{End } \Pi^N \\ w &\longmapsto R(w) \end{aligned}$$

given by

$$\mathcal{R}(w)p(z) = p(zw)$$

for all $z \in \mathbb{C}^N$ and $p \in \Pi^N$.

Lemma 4.1 *For any $u \in \mathbb{C}^N$, $g \in G$, $p \in \Pi^N$, and $z \in \mathbb{C}^N$, we have*

$$\mathcal{R}(g)^{-1} \mathcal{D}_u \mathcal{R}(g)p(z) = \mathcal{D}_{ug}p(z).$$

Proof: The complex Dunkl operator can be written as the sum over the root system R instead of R_+ by

$$\mathcal{D}_u p(z) = \langle \nabla p(z), u \rangle + \frac{1}{2} \sum_{v \in R} \kappa_v \frac{p(z) - p(z\sigma_v)}{\langle z, v \rangle} \langle v, u \rangle.$$

By definition

$$z\sigma_y g = (z\sigma_y)g = zg - 2 \frac{\langle z, y \rangle}{\|y\|^2} yg = (zg)\sigma_{yg},$$

which means

$$\sigma_{yg} = g^{-1} \sigma_y g.$$

Therefore

$$v = yg \implies g\sigma_v = \sigma_y g.$$

Consequently,

$$\begin{aligned} \mathcal{R}(g)\mathcal{D}_{ug}p(z) &= \langle \nabla p(zg), ug \rangle + \frac{1}{2} \sum_{v \in R} \kappa_v \frac{p(zg) - p(zg\sigma_v)}{\langle zg, v \rangle} \langle v, ug \rangle \\ &= \langle \nabla p(zg)g^{-1}, u \rangle + \frac{1}{2} \sum_{y \in R} \kappa_y \frac{p(zg) - p(z\sigma_y g)}{\langle z, y \rangle} \langle y, u \rangle \\ &= \mathcal{D}_u(\mathcal{R}(g)p(z)). \end{aligned}$$

The last step used the fact that

$$\nabla(p(zg)) = (\nabla p(zg))g.$$

□

For $v \in \mathbb{R}^N \setminus \{0\}$, we define the operator ρ_v by

$$\rho_v f(x) = \frac{f(x) - f(x\sigma_v)}{\langle x, v \rangle}.$$

Lemma 4.2 *For any $u \in \mathbb{C}^N$ and $v \in \mathbb{R}^N \setminus \{0\}$,*

$$\langle \nabla, u \rangle \rho_v f(z) - \rho_v \langle \nabla, u \rangle f(z) = \frac{\langle v, u \rangle}{\langle z, u \rangle} \left(\frac{2\langle \nabla f(z\sigma_v), v \rangle}{\langle v, v \rangle} - \frac{f(z) - f(z\sigma_v)}{\langle z, v \rangle} \right).$$

Proof: Notice that

$$\langle \nabla R(\sigma_v)f, u \rangle = \langle R(\sigma_v)\nabla f\sigma_v^{-1}, u \rangle = \langle R(\sigma_v)\nabla f, u\sigma_v \rangle.$$

Therefore

$$\begin{aligned} \langle \nabla, u \rangle \rho_v f(z) &= \frac{\langle \nabla f(z), u \rangle - \langle \nabla f(z\sigma_v), u\sigma_v \rangle}{\langle z, v \rangle} \\ &\quad - \frac{\langle \nabla f(z), u \rangle - (f(z) - f(z\sigma_v))\langle v, u \rangle}{\langle z, v \rangle^2} \end{aligned}$$

and

$$\rho_v \langle \nabla, u \rangle f(z) = \frac{\langle \nabla f(z), u \rangle - \langle \nabla f(z\sigma_v), u \rangle}{\langle z, v \rangle}.$$

By subtracting the above two identities, the result follows from the fact that

$$u - u\sigma_v = 2 \frac{\langle u, v \rangle}{\langle v, v \rangle}.$$

□

5 Main theorem

Our main result gives the commutativity of the complex Dunkl operators.

Theorem 5.1 *Let $R \subset \mathbb{C}^N$ be a root system satisfying Assumption A and let the associated complex Dunkl operators be defined as in (3.1). Then*

$$\mathcal{D}_t \mathcal{D}_u f(z) = \mathcal{D}_u \mathcal{D}_t f(z)$$

for any $f \in \Pi^N$ and $z, t, u \in \mathbb{C}^N$.

Proof: By definition

$$\begin{aligned} \mathcal{D}_t f(z) &= \langle \nabla, t \rangle f(z) + \sum_{v \in R_+} \kappa_v \langle v, t \rangle \rho_v f(z); \\ \mathcal{D}_u f(z) &= \langle \nabla f(z), u \rangle + \sum_{v \in R_+} \kappa_v \langle v, u \rangle \rho_v f(z). \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{D}_t \mathcal{D}_u f(z) &= \langle \nabla, t \rangle \mathcal{D}_u f(z) + \sum_{v \in R_+} \kappa_v \langle v, t \rangle \rho_v \mathcal{D}_u f(z) \\ &= \langle \nabla, t \rangle \left(\langle \nabla, u \rangle f(z) + \sum_{v \in R_+} \kappa_v \langle v, u \rangle \rho_v f(z) \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{v \in R_+} \kappa_v \langle v, t \rangle \rho_v \left(\langle \nabla, u \rangle f(z) + \sum_{v \in R_+} \kappa_v \langle v, u \rangle \rho_v f(z) \right) \\
& = \langle \nabla, t \rangle \langle \nabla, u \rangle f(z) + \sum_{v \in R_+} \kappa_v \langle v, u \rangle \langle \nabla, t \rangle \rho_v f(z) \\
& + \sum_{v \in R_+} \kappa_v \langle v, t \rangle \rho_v \langle \nabla, u \rangle f(z) + \sum_{v, s \in R_+} \kappa_v \kappa_s \langle v, t \rangle \langle s, u \rangle \rho_v \rho_s f(z)
\end{aligned}$$

By symmetricity and notice that

$$\langle \nabla, t \rangle \langle \nabla, u \rangle f(z) = \langle \nabla, u \rangle \langle \nabla, t \rangle f(z),$$

we obtain

$$\begin{aligned}
& \mathcal{D}_t \mathcal{D}_u f(z) - \mathcal{D}_u \mathcal{D}_t f(z) \\
& = \sum_{v \in R_+} \kappa_v \{ \langle v, u \rangle (\langle \nabla, t \rangle \rho_v - \rho_v \langle \nabla, t \rangle) - \langle v, t \rangle (\langle \nabla, u \rangle \rho_v - \rho_v \langle \nabla, u \rangle) \} f(z) \\
& + \sum_{v, s \in R_+} \kappa_v \kappa_s (\langle v, t \rangle \langle s, u \rangle - \langle v, u \rangle \langle s, t \rangle) \rho_v \rho_s f(z).
\end{aligned}$$

The first summand vanishes since Lemma 4.2 implies

$$\begin{aligned}
& \langle v, u \rangle (\langle \nabla, t \rangle \rho_v - \rho_v \langle \nabla, t \rangle) f(z) \\
& = \langle v, u \rangle \frac{\langle v, t \rangle}{\langle z, v \rangle} \left(\frac{2 \langle \nabla f(z \sigma_v), v \rangle}{\langle v, v \rangle} - \frac{f(z) - f(z \sigma_v)}{\langle z, v \rangle} \right) \\
& = \langle v, t \rangle (\langle \nabla, u \rangle \rho_v - \rho_v \langle \nabla, u \rangle) f(z).
\end{aligned}$$

To estimate the second summand, we denote

$$B(v, s) = \langle v, t \rangle \langle s, u \rangle - \langle v, u \rangle \langle s, t \rangle, \quad \forall v, s \in \mathbb{R}^N.$$

It remains to show

$$\sum_{v, s \in R_+} \kappa_v \kappa_s B(v, s) \rho_v \rho_s f(z) = 0. \quad (5.1)$$

Notice that $B(v, s)$ satisfies the following properties:

- (i) $B(v, s)$ is an antisymmetric bilinear form.
- (ii) $B(v\sigma_y, s\sigma_y) = B(s, v), \quad \forall y = av + bs, \quad a, b \in \mathbb{R}, \quad v, s \in R_+.$

Indeed, since $B(v, v) = B(s, s) = B(y, y) = 0$ by antisymmetry, we have

$$\begin{aligned}
 B(v\sigma_y, s\sigma_y) &= B\left(v - 2\frac{\langle v, y \rangle}{\|y\|^2}y, s - 2\frac{\langle s, y \rangle}{\|y\|^2}y\right) \\
 &= B(v, s) - 2\frac{\langle s, y \rangle}{\|y\|^2}bB(v, s) - 2\frac{\langle v, y \rangle}{\|y\|^2}aB(v, s) \\
 &= -B(v, s) = B(s, v).
 \end{aligned}$$

These property imply (5.1) due to Corollary 4.4.7 in [6]. This completes the proof. \square

Example 5.2 *In \mathbb{C}^1 , the root system $R = A_1$ is the unique system satisfying Assumption A. In this case, the reflection group $G = \mathbb{Z}_2$, and the multiplicity function is given by a single parameter $\kappa \in \mathbb{C}$. The complex Dunkl operator is given by*

$$\mathcal{D}f(z) = f'(z) + \kappa \frac{f(z) - f(-z)}{z}, \quad z \in \mathbb{C}.$$

Due to the commutativity of the complex Dunkl operators, we can introduce the related complex Dunkl Laplacian. Let Δ and ∇ be the complex Laplacian and gradient operator

$$\begin{aligned}
 \Delta_{\mathbb{C}} &= \frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2}, \\
 \nabla &= \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right).
 \end{aligned}$$

The complex Dunkl Laplacian is defined as

$$\Delta_h = \mathcal{D}_1^2 + \cdots + \mathcal{D}_N^2.$$

Similar to the real case in [6], we have the formula

$$\Delta_h f(z) = \Delta_{\mathbb{C}} f(z) + 2 \sum_{v \in R_+} \kappa_v \frac{\langle \nabla f(z), v \rangle}{\langle z, v \rangle} - 2 \sum_{v \in R_+} \kappa_v \frac{f(z) - f(\sigma_v z)}{\langle z, v \rangle^2} |v|^2.$$

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